

Null-plane Quantum Universal R -matrix

A. Ballesteros[†], F. J. Herranz[†] and C. M. Pereña[‡]

[†] *Departamento de Física, Universidad de Burgos
E-09001, Burgos, Spain*

[‡] *Departamento de Física Teórica, Universidad Complutense
E-28040, Madrid, Spain*

Abstract

A non-linear map is applied onto the (non-standard) null-plane deformation of (3+1) Poincaré algebra giving rise to a simpler form of this triangular quantization. A universal R -matrix for the null plane quantum algebra is then obtained from a universal T -matrix corresponding to a Hopf subalgebra. Finally, the associated Poincaré Poisson–Lie group is quantized by using the FRT approach.

1 Introduction

In a previous work a new triangular quantum deformation of the (3+1) Poincaré algebra was introduced [1]. This quantization was shown to deform in a consistent way both the kinematical and dynamical contents [2] of the null-plane Poincaré symmetry, obtaining deformed Hamiltonians, spin and position operators. The aim of this letter is to provide a quantum universal R -matrix for this quantum Poincaré algebra that completes this deformed model, including its corresponding quantum group.

We recall that this problem has been recently solved for the (2+1) case by means of a contraction method and taking as starting point the non-standard deformation of $so(2, 2)$ [3, 4]. As we shall see in the sequel, provided a non-linear change of basis inspired in the results given in [4] has been performed, the (3+1) universal R -matrix can be obtained by using a universal T -matrix technique [5, 6]. An interesting consequence of this procedure will be the factorized form of the universal R -matrix, that is given by ordered (usual) exponentials of the elements appearing within the corresponding classical r -matrix. This kind of factorized expressions appears in a natural way in connection with transfer matrices problems in quantum field theory where the R -matrix we obtain could be useful in order to construct new integrable examples.

Let us perform the following transformation on the null-plane generators $\{\tilde{P}_+, \tilde{P}_-, \tilde{E}_i, \tilde{F}_i, \tilde{K}_3, \tilde{J}_3\}$ ($i = 1, 2$) of the quantum Poincaré algebra $U_z\mathcal{P}(3+1)$ [1]:

$$\begin{aligned} P_+ &= \tilde{P}_+, & E_i &= \tilde{E}_i, & J_3 &= \tilde{J}_3, & i &= 1, 2; \\ P_- &= e^{z\tilde{P}_+}\tilde{P}_-, & P_i &= e^{z\tilde{P}_+}\tilde{P}_i, \\ F_1 &= e^{z\tilde{P}_+}(\tilde{F}_1 - z\tilde{E}_1\tilde{P}_- - z\tilde{J}_3\tilde{P}_2), \\ F_2 &= e^{z\tilde{P}_+}(\tilde{F}_2 - z\tilde{E}_2\tilde{P}_- + z\tilde{J}_3\tilde{P}_1), \\ K_3 &= e^{z\tilde{P}_+}(\tilde{K}_3 - z\tilde{E}_1\tilde{P}_1 - z\tilde{E}_2\tilde{P}_2). \end{aligned} \quad (1.1)$$

After this change of basis, the resulting coproduct Δ , non-vanishing commutation relations, counit ϵ and antipode γ read

$$\begin{aligned} \Delta(X) &= 1 \otimes X + X \otimes 1, & \text{for } X &\in \{P_+, E_i, J_3\}, \\ \Delta(Y) &= 1 \otimes Y + Y \otimes e^{2zP_+}, & \text{for } Y &\in \{P_-, P_i\}, \\ \Delta(F_1) &= 1 \otimes F_1 + F_1 \otimes e^{2zP_+} - 2zP_- \otimes E_1e^{2zP_+} - 2zP_2 \otimes J_3e^{2zP_+}, \\ \Delta(F_2) &= 1 \otimes F_2 + F_2 \otimes e^{2zP_+} - 2zP_- \otimes E_2e^{2zP_+} + 2zP_1 \otimes J_3e^{2zP_+}, \\ \Delta(K_3) &= 1 \otimes K_3 + K_3 \otimes e^{2zP_+} - 2zP_1 \otimes E_1e^{2zP_+} - 2zP_2 \otimes E_2e^{2zP_+}; \end{aligned} \quad (1.2)$$

$$\begin{aligned} [K_3, P_+] &= \frac{e^{2zP_+} - 1}{2z}, & [K_3, P_-] &= -P_- - zP_1^2 - zP_2^2, \\ [K_3, E_i] &= E_ie^{2z\tilde{P}_+}, & [K_3, F_i] &= -F_i - 2zK_3P_i, \\ [J_3, P_i] &= -\varepsilon_{ij3}P_j, & [J_3, E_i] &= -\varepsilon_{ij3}E_j, & [J_3, F_i] &= -\varepsilon_{ij3}F_j, \\ [E_i, P_j] &= \delta_{ij}\frac{e^{2zP_+} - 1}{2z}, & [F_i, P_j] &= \delta_{ij}(P_- + zP_1^2 + zP_2^2), \end{aligned}$$

$$\begin{aligned} [E_i, F_j] &= \delta_{ij} K_3 + \varepsilon_{ij3} J_3 e^{2zP_+}, & [P_+, F_i] &= -P_i, \\ [F_1, F_2] &= 2z(P_1 F_2 - P_2 F_1), & [P_-, E_i] &= -P_i; \end{aligned} \quad (1.3)$$

$$\epsilon(X) = 0; \quad \text{for } X \in \{P_\pm, P_i, E_i, F_i, K_3, J_3\}; \quad (1.4)$$

$$\begin{aligned} \gamma(X) &= -X & \text{for } X \in \{P_+, E_i, J_3\}, \\ \gamma(Y) &= -Y e^{-2zP_+} & \text{for } X \in \{P_-, P_i\}, \\ \gamma(F_1) &= -(F_1 + 2zP_- E_1 + 2zP_2 J_3) e^{-2zP_+}, \\ \gamma(F_2) &= -(F_2 + 2zP_- E_2 - 2zP_1 J_3) e^{-2zP_+}, \\ \gamma(K_3) &= -(K_3 + 2zP_1 E_1 + 2zP_2 E_2) e^{-2zP_+}. \end{aligned} \quad (1.5)$$

Note that both the coproduct and commutation relations are now much simpler when compared to the original ones [1]; in particular, the quantum component W_+^q of the Pauli-Lubanski operator has no contribution in (1.3). In general, the map (1.1) can be used to reproduce in this new basis the physically relevant operators introduced in [1]. For instance, the deformed square of the mass M_q^2 is now

$$M_q^2 = P_- \frac{1 - e^{-2zP_+}}{z} - P_1^2 e^{-2zP_+} - P_2^2 e^{-2zP_+}, \quad (1.6)$$

and it induces a deformed null-plane evolution governed by a q -Schrödinger equation that has been studied for the (2+1) case in [4].

The key of our construction of the universal R -matrix is to focus on the six generators appearing in the classical r -matrix underlying this quantum deformation

$$r = 2(K_3 \wedge P_+ + E_1 \wedge P_1 + E_2 \wedge P_2), \quad (1.7)$$

since they close a Hopf subalgebra $U_z g$ after quantization. The universal T -matrix for this Hopf subalgebra can be computed, and this canonical element will give rise to a (factorized) universal R -matrix for $U_z g$ in a straightforward way. The important point is that this R -matrix can be shown to be a universal R -matrix for the whole null-plane quantum Poincaré algebra. As a first application of this result, the null-plane quantum Poincaré group will be obtained.

2 Universal T -matrix for $U_z g$

The universal T -matrix of a Hopf algebra, considered for the first time by Fronsdaal and Galindo [5, 6], is the Hopf algebra dual form

$$T = \sum_{\mu} X^{\mu} \otimes p_{\mu}, \quad (2.1)$$

where $\{X^{\mu}\}$ is a basis for the Hopf algebra and $\{p_{\mu}\}$ its dual: $\langle p_{\nu}, X^{\mu} \rangle = \delta_{\nu}^{\mu}$. Let us remark that, in spite of the presence of a particular basis in (2.1), the T -matrix is by definition basis-independent.

We are interested in the Hopf algebra dual form for $U_z g$ with generators $\{P_+, P_i, E_i, K_3\}$ ($i = 1, 2$). Let us choose the basis $X^{abcdef} = E_2^a E_1^b P_+^c K_3^d P_1^e P_2^f$. Its dual basis will be given by the monomials p_{lmnpqr} such that

$$\langle p_{lmnpqr}, X^{abcdef} \rangle = \delta_l^a \delta_m^b \delta_n^c \delta_p^d \delta_q^e \delta_r^f. \quad (2.2)$$

We can express duality in an explicit way by means of two structure tensors E and F that give, respectively, the product and the coproduct in the Hopf algebra. For our purposes it suffices to consider the latter, so we have

$$\Delta(X^{abcdef}) := F_{ijklmn;pqrstu}^{abcdef} X^{ijklmn} \otimes X^{pqrstu} \quad (2.3)$$

$$p_{ijklmn} p_{pqrstu} = F_{ijklmn;pqrstu}^{abcdef} p_{abcdef}. \quad (2.4)$$

In order to compute the T -matrix we only need to know very few selected components of this tensor. This fact has been already used in [7], where all the essential reasonings needed to prove the following statements can be found. From (2.4) and taking into account that $p_{000000} = 1$ we get

$$\begin{aligned} F_{000000;pqrstu}^{abcdef} &= \delta_p^a \delta_q^b \delta_r^c \delta_s^d \delta_t^e \delta_u^f \\ F_{ijklmn;000000}^{abcdef} &= \delta_i^a \delta_j^b \delta_k^c \delta_l^d \delta_m^e \delta_n^f \\ F_{ijklmn;pqrstu}^{000000} &= \delta_i^0 \delta_j^0 \delta_k^0 \delta_l^0 \delta_m^0 \delta_n^0 \delta_p^0 \delta_q^0 \delta_r^0 \delta_s^0 \delta_t^0 \delta_u^0. \end{aligned} \quad (2.5)$$

And by computing the coproducts of some specific monomials X^{abcdef} and comparing them to (2.3), it can be checked that

$$\begin{aligned} F_{100000;pqrstu}^{abcdef} &= a \delta_{p+1}^a \delta_q^b \delta_r^c \delta_s^d \delta_t^e \delta_u^f \\ F_{010000;0qrstu}^{abcdef} &= b \delta_0^a \delta_{q+1}^b \delta_r^c \delta_s^d \delta_t^e \delta_u^f \\ F_{001000;00rstu}^{abcdef} &= c \delta_0^a \delta_0^b \delta_{r+1}^c \delta_s^d \delta_t^e \delta_u^f \\ F_{000lmn;000001}^{abcdef} &= f \delta_0^a \delta_0^b \delta_0^c \delta_l^d \delta_m^e \delta_{n+1}^f \\ F_{000lm0;000010}^{abcdef} &= e \delta_0^a \delta_0^b \delta_0^c \delta_l^d \delta_{m+1}^e \delta_0^f \\ F_{000l00;000100}^{abcdef} &= d \delta_0^a \delta_0^b \delta_0^c \delta_{l+1}^d \delta_0^e \delta_0^f. \end{aligned} \quad (2.6)$$

From all of them, and if we denote

$$\begin{aligned} \hat{e}_2 &= p_{100000} & \hat{e}_1 &= p_{010000} & \hat{a}_+ &= p_{001000} \\ \hat{k}_3 &= p_{000100} & \hat{a}_1 &= p_{000010} & \hat{a}_2 &= p_{000001} \end{aligned} \quad (2.7)$$

the following expression for the dual basis is derived

$$p_{lmnpqr} = \frac{\hat{e}_2^l \hat{e}_1^m \hat{a}_+^n \hat{k}_3^p \hat{a}_1^q \hat{a}_2^r}{l! m! n! p! q! r!}. \quad (2.8)$$

Then from (2.1) we get the final expression for the $U_z g$ dual form

$$\mathcal{T} = e^{E_2 \otimes \hat{e}_2} e^{E_1 \otimes \hat{e}_1} e^{P_+ \otimes \hat{a}_+} e^{K_3 \otimes \hat{k}_3} e^{P_1 \otimes \hat{a}_1} e^{P_2 \otimes \hat{a}_2}. \quad (2.9)$$

3 Universal R -matrix

A universal R -matrix for $U_z g$ can be easily deduced from the T -matrix (2.9) provided there exists an algebra homomorphism and coalgebra antihomomorphism Φ between the quantum algebra $U_z g$ and its associated dual Hopf algebra (quantum group) $Fun_z(G)$ [8]. If this condition is fulfilled, the element

$$\mathcal{R} = (\text{id} \otimes \Phi)\mathcal{T}, \quad (3.1)$$

with Φ acting on the generators of $Fun_z(G)$, is a solution of the quantum Yang–Baxter equation and, moreover,

$$\mathcal{R}\Delta(X)\mathcal{R}^{-1} = \sigma \circ \Delta(X), \quad \text{with} \quad \sigma(a \otimes b) = b \otimes a. \quad (3.2)$$

In our case, let us compute the defining relations for $Fun_z(G)$. The elements in (2.7) are precisely dual to the generators of $U_z g$, so they can be taken as generators for $Fun_z(G)$. The main tool in order to derive the commutation rules among them is again the structure tensor F . From (2.4) we have

$$[p_{ijklmn}, p_{pqrstu}] = (F_{ijklmn;pqrstu}^{abcdef} - F_{pqrstu;ijklmn}^{abcdef})p_{abcdef} \quad (3.3)$$

for two arbitrary elements in $Fun_z(G)$. Let us recall that we can obtain information about the components of the tensor F from the coproduct in the quantum algebra, exactly as we did in the previous section. In particular, the relevant terms in order to compute $[x, y]$, where x and y are two of the generators considered above, are $X \otimes Y$ and $Y \otimes X$ (X and Y being their respective dual elements) appearing in the coproduct of elements of $U_z g$ [7]. This, together with a careful preservation of the order throughout the computations allow us to derive the following relations

$$[\hat{k}_3, \hat{a}_+] = 2z(e^{\hat{k}_3} - 1), \quad [\hat{a}_i, \hat{a}_+] = 2z\hat{a}_i e^{\hat{k}_3}, \quad [\hat{e}_i, \hat{a}_j] = 2z\delta_{ij}(e^{\hat{k}_3} - 1), \quad (3.4)$$

and to show that the remaining commutators vanish. In order to find the coproduct of $Fun_z(G)$ we consider a 5×5 matrix representation of $U_z g$:

$$\begin{aligned} D(P_+) &= \frac{1}{2}(e_{10} + e_{40}), & D(P_1) &= e_{20}, & D(P_2) &= e_{30}, & D(K_3) &= e_{14} + e_{41}, \\ D(E_1) &= \frac{1}{2}(e_{12} + e_{21} - e_{24} + e_{42}), & D(E_2) &= \frac{1}{2}(e_{13} + e_{31} - e_{34} + e_{43}), \end{aligned} \quad (3.5)$$

where e_{ij} is the matrix with a single 1 entry at row i , column j , and zeros at the remaining entries (note that this representation is a classical one). By applying (3.5) in the universal T -matrix (2.9) we obtain an element of the quantum group $Fun_z(G)$:

$$\begin{aligned} D(\mathcal{T}) &= \exp\{D(E_2)\hat{e}_2\} \exp\{D(E_1)\hat{e}_1\} \exp\{D(P_+)\hat{a}_+\} \\ &\quad \times \exp\{D(K_3)\hat{k}_3\} \exp\{D(P_1)\hat{a}_1\} \exp\{D(P_2)\hat{a}_2\} \end{aligned}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(\hat{a}_+ + \hat{e}_1\hat{a}_1 + \hat{e}_2\hat{a}_2) & \cosh(\hat{k}_3) + f & \frac{1}{2}\hat{e}_1 & \frac{1}{2}\hat{e}_2 & \sinh(\hat{k}_3) - f \\ \hat{a}_1 & \frac{1}{2}\hat{e}_1 e^{-\hat{k}_3} & 1 & 0 & -\frac{1}{2}\hat{e}_1 e^{-\hat{k}_3} \\ \hat{a}_2 & \frac{1}{2}\hat{e}_2 e^{-\hat{k}_3} & 0 & 1 & -\frac{1}{2}\hat{e}_2 e^{-\hat{k}_3} \\ \frac{1}{2}(\hat{a}_+ + \hat{e}_1\hat{a}_1 + \hat{e}_2\hat{a}_2) & \sinh(\hat{k}_3) + f & \frac{1}{2}\hat{e}_1 & \frac{1}{2}\hat{e}_2 & \cosh(\hat{k}_3) - f \end{pmatrix} \quad (3.6)$$

where

$$f = \frac{1}{8}(\hat{e}_1^2 + \hat{e}_2^2) e^{-\hat{k}_3}. \quad (3.7)$$

Hence, the coproduct of $Fun_z(G)$ can be derived from the multiplication of two quantum matrices $D(\mathcal{T}) \dot{\otimes} D(\mathcal{T})$:

$$\begin{aligned} \Delta(\hat{k}_3) &= 1 \otimes \hat{k}_3 + \hat{k}_3 \otimes 1, & \Delta(\hat{a}_i) &= 1 \otimes \hat{a}_i + \hat{a}_i \otimes 1, \\ \Delta(\hat{e}_i) &= e^{\hat{k}_3} \otimes \hat{e}_i + \hat{e}_i \otimes 1, \\ \Delta(\hat{a}_+) &= e^{\hat{k}_3} \otimes \hat{a}_+ + \hat{a}_+ \otimes 1 - \hat{a}_1 e^{\hat{k}_3} \otimes \hat{e}_1 - \hat{a}_2 e^{\hat{k}_3} \otimes \hat{e}_2. \end{aligned} \quad (3.8)$$

By recalling the coproduct and commutation rules of $U_z g$ given in (1.2) and (1.3), together with the expressions (3.8) and (3.4), we get the map we were looking for:

$$\Phi(\hat{a}_+) = -2zK_3, \quad \Phi(\hat{a}_i) = -2zE_i, \quad \Phi(\hat{k}_3) = 2zP_+, \quad \Phi(\hat{e}_i) = 2zP_i. \quad (3.9)$$

Hence, the universal R -matrix for $U_z g$ is:

$$\begin{aligned} \mathcal{R} &= \exp\{2zE_2 \otimes P_2\} \exp\{2zE_1 \otimes P_1\} \exp\{-2zP_+ \otimes K_3\} \\ &\quad \times \exp\{2zK_3 \otimes P_+\} \exp\{-2zP_1 \otimes E_1\} \exp\{-2zP_2 \otimes E_2\}. \end{aligned} \quad (3.10)$$

The T -matrix construction ensures that the element (3.10) is a solution of the quantum Yang–Baxter equation and that (3.2) is fulfilled for all the generators of the Hopf subalgebra. Furthermore, this condition is also true for the four remaining generators (P_- , F_1 , F_2 and J_3). This fact can be proved by computing $\mathcal{R} \Delta \mathcal{R}^{-1}$ for each of them, with \mathcal{R} written in the form $\mathcal{R} = e^{A_1} e^{A_2}$ where

$$A_1 = 2z(E_1 \otimes P_1 + E_2 \otimes P_2 - P_+ \otimes K_3), \quad A_2 = -2z(P_1 \otimes E_1 + P_2 \otimes E_2 - K_3 \otimes P_+). \quad (3.11)$$

Therefore, we conclude that (3.10) is a universal R -matrix for the null-plane deformation of Poincaré algebra (1.2)–(1.5). This R -matrix is just the natural generalization of the (2+1) result given in [4], and can be seen as an ordered exponentiation of the classical r -matrix (1.7).

4 Null-plane quantum Poincaré group

The following generic element of the quantum Poincaré group $Fun_z(P(3+1))$:

$$D(\mathcal{P}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{\hat{x}^+}{2} + \hat{x}^- & \hat{\Lambda}_0^0 & \hat{\Lambda}_1^0 & \hat{\Lambda}_2^0 & \hat{\Lambda}_3^0 \\ \hat{x}^1 & \hat{\Lambda}_0^1 & \hat{\Lambda}_1^1 & \hat{\Lambda}_2^1 & \hat{\Lambda}_3^1 \\ \hat{x}^2 & \hat{\Lambda}_0^2 & \hat{\Lambda}_1^2 & \hat{\Lambda}_2^2 & \hat{\Lambda}_3^2 \\ \frac{\hat{x}^+}{2} - \hat{x}^- & \hat{\Lambda}_0^3 & \hat{\Lambda}_1^3 & \hat{\Lambda}_2^3 & \hat{\Lambda}_3^3 \end{pmatrix}, \quad (4.1)$$

can be regarded as the non-commutative analogue of a null-plane Poincaré group element obtained by means of a 5×5 matrix representation of the quantum Poincaré algebra given by the six matrices (3.5) together with

$$\begin{aligned} D(P_-) &= e_{10} - e_{40}, & D(F_1) &= e_{12} + e_{21} + e_{24} - e_{42}, \\ D(J_3) &= e_{23} - e_{32}, & D(F_2) &= e_{13} + e_{31} + e_{34} - e_{43}, \end{aligned} \quad (4.2)$$

where the exponentials of the translation generators P_\pm , P_i have been located at the left. The quantum Lorentz coordinates $\hat{\Lambda}_\nu^\mu$ satisfy the pseudo-orthogonality condition:

$$\hat{\Lambda}_\nu^\mu \hat{\Lambda}_\sigma^\rho \eta^{\nu\sigma} = \eta^{\mu\rho}, \quad (\eta^{\mu\rho}) = \text{diag}(1, -1, -1, -1). \quad (4.3)$$

The coproduct of $Fun_z(P(3+1))$ is provided by $D(\mathcal{P}) \dot{\otimes} D(\mathcal{P})$. The counit and antipode come from relations $\epsilon(D(\mathcal{P})) = I$ (I is the 5×5 identity matrix) and $\gamma(D(\mathcal{P})) = D(\mathcal{P})^{-1}$. On the other hand, the associated commutation relations of the quantum Poincaré group can be deduced via the FRT approach [9]. The universal R -matrix (3.10) written in the representation (3.5) reduces to

$$D(\mathcal{R}) = I \otimes I + 2z(D(K_3) \wedge D(P_+) + D(E_1) \wedge D(P_1) + D(E_2) \wedge D(P_2)). \quad (4.4)$$

Since this element fulfills the property (3.2) we can apply the prescription

$$D(\mathcal{R})D(\mathcal{P})_1 D(\mathcal{P})_2 = D(\mathcal{P})_2 D(\mathcal{P})_1 D(\mathcal{R}), \quad (4.5)$$

where $D(\mathcal{P})_1 = D(\mathcal{P}) \otimes I$ and $D(\mathcal{P})_2 = I \otimes D(\mathcal{P})$, thus obtaining the commutation rules of $Fun_z(P(3+1))$:

$$\begin{aligned} [\hat{x}^+, \hat{x}^i] &= -2z \hat{x}^i, & [\hat{x}^+, \hat{x}^-] &= -2z \hat{x}^-, & [\hat{x}^i, \hat{x}^-] &= 0, & [\hat{x}^1, \hat{x}^2] &= 0, \\ [\hat{\Lambda}_\nu^\mu, \hat{\Lambda}_\sigma^\rho] &= 0, & \nu, \mu, \rho, \sigma &= 0, 1, 2, 3; \\ [\hat{\Lambda}_\nu^\mu, \hat{x}^+] &= -2z \delta_{\mu 0} (\hat{\Lambda}_\nu^3 - \delta_{\nu 0} + \delta_{\nu 3}) - 2z \delta_{\mu 3} (\hat{\Lambda}_\nu^0 + \delta_{\nu 0} - \delta_{\nu 3}) \\ &\quad + z (\hat{\Lambda}_0^\mu + \hat{\Lambda}_3^\mu) (\hat{\Lambda}_\nu^0 + \hat{\Lambda}_\nu^3), \\ [\hat{\Lambda}_\nu^\mu, \hat{x}^-] &= \frac{1}{2} z \delta_{\mu 0} (-\delta_{\nu 0} + \delta_{\nu 3}) + \frac{1}{2} z \delta_{\mu 3} (-\delta_{\nu 0} + \delta_{\nu 3}) + \frac{1}{2} z (\hat{\Lambda}_0^\mu + \hat{\Lambda}_3^\mu) (\hat{\Lambda}_\nu^0 - \hat{\Lambda}_\nu^3), \\ [\hat{\Lambda}_\nu^\mu, \hat{x}^1] &= z \delta_{\mu 2} \hat{\Lambda}_\nu^1 + z \delta_{\mu 1} (-\hat{\Lambda}_\nu^0 + \hat{\Lambda}_\nu^1 + \hat{\Lambda}_\nu^3 + \delta_{\nu 0} - \delta_{\nu 3}) + z \hat{\Lambda}_\nu^1 (\hat{\Lambda}_0^\mu + \hat{\Lambda}_3^\mu - 1), \\ [\hat{\Lambda}_\nu^\mu, \hat{x}^2] &= z \delta_{\mu 1} \hat{\Lambda}_\nu^2 + z \delta_{\mu 2} (-\hat{\Lambda}_\nu^0 + \hat{\Lambda}_\nu^2 + \hat{\Lambda}_\nu^3 + \delta_{\nu 0} - \delta_{\nu 3}) + z \hat{\Lambda}_\nu^2 (\hat{\Lambda}_0^\mu + \hat{\Lambda}_3^\mu - 1). \end{aligned} \quad (4.6)$$

The commutators among the quantum coordinates \hat{x}^\pm , \hat{x}^i can be interpreted as the null-plane quantum Poincaré plane. It is also worth mentioning that the universal R -matrix can be used to construct a q -differential calculus [10] on the null-plane quantum Poincaré group; in general, non-standard deformations exhibit interesting properties in this context [11].

It can be checked these commutators are a Weyl quantization of the Poisson brackets of the coordinate functions $\{x^\pm, x^i, \Lambda_\nu^\mu\}$ on the classical Poincaré group, which can be obtained by means of the Poisson bivector

$$\{D(\mathcal{P}) \dot{\otimes} D(\mathcal{P})\} = [r, D(\mathcal{P}) \dot{\otimes} D(\mathcal{P})], \quad (4.7)$$

writing the classical r -matrix (1.7) in terms of the matrix representation (3.5) and applying the corresponding pseudo-orthogonality relations (4.3). This was the method used in [12, 13] to construct the κ -Poincaré group (recall that, in this case, only the (2+1) universal R -matrix has been found in [14]).

Acknowledgements

This work has been partially supported by DGICYT de España (Projects PB92–0197 and PB94–1115). CMP is grateful to the Departamento de Física in Burgos for their hospitality.

References

- [1] A. Ballesteros, F.J. Herranz, M.A. del Olmo and M. Santander, Phys. Lett. B 351 (1995) 137.
- [2] H. Leutwyler and J. Stern, Ann. Phys. (N.Y.) 112 (1978) 94.
- [3] A. Shariati, A. Aghamohammadi and M. Khorrami, Mod. Phys. Lett. A 11 (1996) 187.
- [4] A. Ballesteros and F.J. Herranz, Mod. Phys. Lett. A to appear (q-alg/9605031).
- [5] C. Fronsdal and A. Galindo, Lett. Math. Phys. 27 (1993) 39.
- [6] C. Fronsdal and A. Galindo “*The Universal T -Matrix*” in: Proc. of the Joint Summer Research Conference on conformal field theory, topological field theory and quantum groups. Holyhoke, 1992.
- [7] A. Ballesteros, F.J. Herranz, M.A. del Olmo, C.M. Pereña and M. Santander, J. Phys. A 28 (1995) 7113.
- [8] C. Fronsdal, “*Universal T -Matrix for Twisted Quantum $gl(N)$* ” preprint UCLA/93/TEP/3.
- [9] N.Y. Reshetikhin, L.A. Takhtadzhyan and L.D. Faddeev, Leningrad Math. J. 1 (1990) 193.
- [10] P. Aschieri and L. Castellani, Int. J. Mod. Phys. A 8 (1993) 1667.
- [11] A. Aghamohammadi, M. Khorrami and A. Shariati, J. Phys. A 28 (1995) L225.
- [12] P. Maślanka, J. Phys. A 26 (1993) L1251.
- [13] J. Lukierski and H. Ruegg, Phys. Lett. B 329 (1994) 189.
- [14] E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Math. Phys. 32 (1991) 1159.